Boundary face method for 3D contact problems with non-conforming contact discretization

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ABSTRACT

Three-dimensional contact problems without friction have been studied using the Boundary Face Method (BFM). In this paper, a non-conforming contact discretization approach is used to enforce the contact conditions between the two contact surfaces. This method is based on Node-To-Surface (NTS), and there is no need that the identical discretization is performed along the contact surfaces of both bodies. The contact equations are written explicitly with both tractions and displacements which are retained as unknowns in Boundary Integral Equation (BIE). An iterative procedure is presented to determine the correct contact zone by obtaining a solution compatible with the contact conditions (no interpenetrations between the domains and no tensile on the final contact zone). Several numerical examples have been presented to illustrate the applicability of the method.

Keywords: contact problems; boundary face method; non-conforming contact discretization;

1. Introduction

Boundary value problems involving contact are of great importance in industry related to mechanical and civil engineering. The load transferred through a mechanical assemblage usually causes stress concentrations, which increase the risk for crack initiations, propagations and fatigue failure. Contact problems are complex and inherently nonlinear due to their moving boundaries and the friction along contact surfaces if it is involved. When solid bodies are brought, they will either be initially in contact at a point, along a line, or over a surface. Under the action of external loads, the contact area changes progressively and in most cases the extent of contact is *a priori* unknown, which presents a boundary non-linearity. Even when friction is present between the contacting surfaces, the problem becomes even more difficult because the contact area may exhibit adhesion and slip zones which are of unknown extent and often of a more complicated form than the contact area itself. In a word, the solution of the contact problems will depend on the magnitude of the external load, the material properties of the contacting bodies and the frictional behavior at the contact interfaces.

Since Hertz [1] published his famous work on normal contact between elastic bodies, many papers has been published giving analytical solutions. An excellent review of all these work can be found in Gladwell [2] and Johnson [3]. However, a common feature in these studies is that the geometries (infinite or semi-finite domains), properties of the materials (isotropic) and characterization of the contact (frictionless) are assumed, so that available mathematical and mechanical tools can be used to obtain a closed-form solution of the problem. Obviously, these approaches are very restrictive and can only be applied to several special problems.

With the rapidly increasing power of modern computers, more and more efforts have been devoted to improve the numerical treatment of contact problems. Numerical methods mainly based on the finite element method (FEM), boundary element method (BEM) and meshless methods have been applied to try to develop a general tool of analysis. The FEM has been widely used to analyze the contact problems in many engineering examples. An excellent review of FEM in contact mechanics can be found in the text books by Zhong [4], Laursen [5] and Wriggers[6], and as the chapters in the FEM text books, for example, Bathe [7], Crisfield [8], Belytschko [9], and Zienkiewicz and Taylor [10]. The BEM is regarded as an important complementary method to FEM. The BEM appears to be very suitable to deal with the contact problems when the non-linearity only occurs at the contact boundary.

The first application of BEM to contact problems can be traced back to Andersson [11] in 1980. Constant elements were employed and only the frictionless contact problems were considered. Later, the friction and linear and quadratic elements were introduced in his next work [12]. Paris et al. [13] applied BEM to solve two-dimensional contact problems using discontinuous elements. The nonlinear friction law was introduced to study two-dimensional frictional contact problems with BEM by Jin et al. [14]. Dandekar and Conant [15-17] presented their work and proposed an efficient equation solver for two-dimensional frictional contact problems. For three-dimensional problems, Paris et al. [18] did the first work for frictionless contact problems, and then Garrido et al. [19] expanded their work to solve the frictional contact problems. Leahy and Becker [20,21] developed the three-dimensional frictional contact problems using a quadratic boundary element formulation. Segond and Tafreshi [22] presented their work to implement the frictionless and infinite friction conditions in contact problems. All the mentioned above, these approaches imposed the contact constraints directly and written the contact equations explicitly. Meanwhile, other formulations to treat the contact conditions also can be found. Takahashi [23] proposed the flexibility approach to analyze the two-dimensional contact problems between an elastic body and a rigid body using constant element, and expanded this approach to study contact problems between elastic bodies with friction. Yamazaki et al. [24] published their approaches based on penalty parameter method. The same as FEM, the mathematical programming approach was also introduced to analyze frictional contact problems with BEM [25-27]. All the work mentioned above, the conforming contact discretization approach has been used, which requires the identical discretization in the contact area. The contact conditions are imposed by means of node to node sequence.

Blazquez et al. [29-31] did the first work using the non-conforming contact discretization to study the two-dimensional frictionless contact problems, and expanded this approach to the frictional contact problems. Paris et al. [32] also presented their same work and suggested using linear discontinuous elements for nonconforming contact discretization in two-dimensional contact problems. Martin and Aliabadi [33] proposed a BE hyper-singular formulation for two-dimensional contact problems using non-conforming contact discretization. However, all these works have been implemented to solve two-dimensional contact problems.

The boundary face method (BFM) has been developed by Zhang *et al*. [37] which is also based on the boundary integral equation. However, both boundary integration and variable approximation in BFM are performed on boundary faces, which are represented in parametric form exactly as the boundary representation data structure in most CAD systems. Later, many implements based on the BFM can be found [38-43].

Therefore, the same as BEM, there are three main advantages for using BFM when studying contact problems:

(1) Only the boundaries of the contacting bodies need to be discretized, which are primary interest in solution procedure. It is not necessary to compute the internal stress and displacement.

(2) The contact equations are written explicitly with both traction and displacement which are independent variables that appear in BEM. It is also able to couple the normal and tangential tractions directly in the system equations when the friction is present.

(3) The contact pressure which can be obtained directly from the traction distribution, is a primarily unknown quantity, and is solved with the same accuracy as the displacement unknown.

In this paper, the BFM will be used to solve the three-dimensional frictionless contact problems with non-conforming contact discretization approach. So, there is no need that the identical discretization is performed along the contact surfaces of both bodies. An iterative procedure is presented to determine the correct contact zone by obtaining a solution compatible with the contact conditions (no interpenetrations between the domains and no tensile on the final contact zone). Although the friction appears naturally in all contact problems, from a numerical point of view, our work in this paper represents a previous development to the analysis of three-dimensional frictional contact problems. The rest of this article is organized as follows: in Section 2, we introduce a general description of frictionless contact problem. In Section 3, the boundary integral equation for contact problem is introduced, and the definition of contact conditions with non-conforming contact discretization is presented in Section 4. In Section 5, the solution procedure is developed. Numerical examples are presented to illustrate the applicability of the algorithm in Section 6. Finally, the paper ends with conclusions in Section 7.

2. Definition of the frictionless contact problem

In this paper we consider two linear elastic bodies A and B which occupy respectively the domains $\boldsymbol{D}^{\boldsymbol{A}}$ and $\boldsymbol{D}^{\boldsymbol{B}}$ in $\boldsymbol{R}^{\mathbf{3}}$. The boundaries of the two domains are defined as S^A and S^B. Two complementary zones S_C and S_N will be defined in each of the boundaries. The zone S_C is the part of the boundary which makes contact with the other domain, and the zone S_N is the part of the boundary where the contact does not occur. Therefore

$$S^K = S_C \cup S_N^K, \quad (K = A, B)$$

The non-contact zone S_N^K is considered subdivided into three zones depending on the boundary conditions on them. The displacements are known on S_{NU}^K, the stress vectors are known on S_{NT}^K and the mixed conditions of displacements and stresses are prescribed on S_{NUT}^K. Therefore

$$S_N^K = S_{NU}^K \cup S_{NT}^K \cup S_{NUT}^K, \quad (K = A, B)$$

As a result, the boundary of each body can be considered as consisting of nodes outside the contact zone and nodes inside the contact zone which will be referred as contact nodes in this paper. Outside the contact zone either displacements or tractions (but not both) are known at every boundary node. Inside the contact zone neither displacements nor tractions are known. Consider two bodies (***A*** and ***B***), the contact zone S_C, is formed between these bodies due to the application of external forces as shown in Fig.1.

In contact problems, unilateral conditions [44] hold, namely,

(1) No material interpenetration.

(2) The normal component of traction is compressive.

(3) The complementary condition must be satisfied.

For convenience, let P^A denote a point on body A and P^B a point on body B. Suppose the two points are within the potential contact zone and that the tangent and normal vectors at the two points are shown in Fig.2. These conditions can be expressed by the following equations (with reference to Fig.2):

$$g = u_1^A(x) + u_1^B(x) - \delta_1 \le 0 \tag{1}$$

$$p = t_1^A(x) = t_1^B(x) \le 0 \tag{2}$$

$$pg = 0 \tag{3}$$

where g is contact gap, δ_1 is the initial normal distance between the two points and p is contact pressure (negative normal tractions). For frictionless problems, the tangent tractions satisfy the following equation:

$$t_2^A(x) = t_2^B(x) = t_3^A(x) = t_3^B(x) = 0 \tag{4}$$

where the subscripts *1*, *2* , *3* refer to the axes shown in Fig.2.

Therefore, the local coordinate system has to be introduced to establish the contact equations. For convenience, take two-dimension problems for example as shown in Fig.3. The BFM can accurately obtain the unique normal on the node P between the elements S^1 and S^2 which is directly calculated by the parametric form of the boundary. Nevertheless, the contact normal on the node P is not unique which is calculated by the elements S^1 and S^2 in conventional BEM/FEM. This is the advantage of BFM over the BEM and FEM.

In the absence of friction the elastic contact problem is path independent, and the total quantities can be used to solve the contact problem. An iterative procedure is used to find the final contact zone since no energy is dissipating in the contact zone.

3. Boundary integral equation for contact problem

In general, the boundary integral equation for each elastic contacting body can be written in the following form with the absence of body forces

$$\begin{aligned}&C_{ij}u_j(P)+\int_S T_{ij}(P,Q)u_j(Q)dS(Q)=\int_S U_{ij}(P,Q)t_j(Q)dS(Q)\\&i,j=1,2,3\quad P,Q\in S\end{aligned} \tag{5}$$

where $u_j(Q)$ and $t_j(Q)$ are the displacements and stress vectors at points on the boundary and $T_{ij}(P,Q)$ and $U_{ij}(P,Q)$ are fundamental solutions and can be given as following for the three-dimensional case

$$U_{ij}(P,Q)=\frac{1}{16\pi G(1-\nu)r}[(3-4\nu)\delta_{ij}+r_{,i}r_{,j}] \tag{6}$$

$$T_{ij}(P,Q)=\frac{1}{8\pi(1-\nu)r^2}[\frac{\partial r}{\partial n}((1-2\nu)\delta_{ij}+r_{,i}r_{,j})+(1-2\nu)(n_jr_{,i}-n_ir_{,j})] \tag{7}$$

where G and v are the elastic constants of the material, r represents the positional vector with its origin at point P and its end at point Q, and n is the outward normal to the boundary.

Discretizing the boundary using the quadrilateral elements, the discretized format of Eq. (5) can be obtained as

$$\begin{aligned}&C_{ij}u_j(P)+\sum_{k=1}^{N^e}\left\{\sum_{l=1}^{M}u_j^{k,l}\int_{S^k}T_{ij}(P,Q(\xi,\eta))N_l(\xi,\eta)\left|J(\xi,\eta)\right|dS(\xi,\eta)\right\}\\&=\sum_{k=1}^{N^e}\left\{\sum_{l=1}^{M}t_j^{k,l}\int_{S^k}U_{ij}(P,Q(\xi,\eta))N_l(\xi,\eta)\left|J(\xi,\eta)\right|dS(\xi,\eta)\right\}\end{aligned} \tag{8}$$

where S^k represents the kth boundary element, N^e represents the total number of elements, M represents the number of nodes per element, $u_j^{k,l}$ and $t_j^{k,l}$ are the displacement and traction components of the lth node of the kth element, respectively. In BFM, the numerical integrations of the boundary element in Eq. (8) are implemented in the parametric space of the boundary surfaces, where the geometric data at Gaussian quadrature points in integration elements, such as the coordinates, the Jacobian and the outward normal are calculated directly from the surface rather than elements, thus no geometric error will be introduced.

With the collocation point P being placed at all nodes, the matrix from of Eq. (8) can be written as

$$\mathbf{Hu} = \mathbf{Gt} \tag{9}$$

Considering the boundary nodes which is in contact or not in the contact problems, Eq. (9) can be written as

$$\begin{bmatrix} \mathbf{H}_{NN} & \mathbf{H}_{NC} \\ \mathbf{H}_{CN} & \mathbf{H}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_N \\ \mathbf{u}_C \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{NN} & \mathbf{G}_{NC} \\ \mathbf{G}_{CN} & \mathbf{G}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_N \\ \mathbf{t}_C \end{Bmatrix} \tag{10}$$

Considering the boundary and contact conditions, the displacements and tractions are unknown for the nodes in the contact zone, Eq. (10) is written as

$$\begin{bmatrix} \mathbf{A}_{NN} & \mathbf{H}_{NC} & \mathbf{G}_{NC} \\ \mathbf{A}_{CN} & \mathbf{H}_{CC} & \mathbf{G}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_N \\ \mathbf{u}_C \\ \mathbf{t}_C \end{Bmatrix} = \bar{\mathbf{b}} \tag{11}$$

where N denotes the non-contact nodes and C denotes the contact nodes; $\mathbf{x}_N$ is the unknowns at the non-contact nodes with boundary conditions.

For both contact bodies A and B, Eq. (11) can be assembled. Therefore, the system matrix can be assembled as

$$\begin{bmatrix} \mathbf{A}_{NN}^A & \mathbf{H}_{NC}^A & \mathbf{G}_{NC}^A & & & \\ \mathbf{A}_{CN}^A & \mathbf{H}_{CC}^A & \mathbf{G}_{CC}^A & & & \\ & & & \mathbf{A}_{NN}^B & \mathbf{H}_{NC}^B & \mathbf{G}_{NC}^B \\ & & & \mathbf{A}_{CN}^B & \mathbf{H}_{CC}^B & \mathbf{G}_{CC}^B \\ & & C^A & & & \\ & & & & C^B & \end{bmatrix} \begin{Bmatrix} \mathbf{x}_N^A \\ \mathbf{u}_C^A \\ \mathbf{t}_C^A \\ \mathbf{x}_N^B \\ \mathbf{u}_C^B \\ \mathbf{t}_C^B \end{Bmatrix} = \mathbf{f} \tag{12}$$

where the C^A and C^B denote the contact equations for the contact nodes on the contact body A and B, respectively.

In Eq. (12), three algebraic equations can be obtained for each collocation node of the contacting bodies by the Boundary Integral Equation (BIE). Considering the boundary and contact conditions, there are three unknowns (three displacement components or three traction components or mixed displacement and traction components) for the nodes outside the contact zone; there are six unknowns (three displacement components and three traction components) for nodes inside the contact zone. As a result, the number of algebraic equation is less than the number of boundary unknowns. So, three additional equations must be complemented for each node inside the contact zone, according to the contact conditions which will be shown in the next section.

4. Definition of contact conditions with non-conforming contact discretization

An antecedent of the non-conforming contact discretization approach can be found in Hallquist [35] using finite element method. This contact model is usually called as 'node-to-segment' in 2D or 'node-to-surface' in 3D in certain research papers.

In this approach, the shape functions are used to establish the contact constraints between a contact node of one body and a boundary element (target element) of the other where the target element is obtained by the closest point projection. Then contact constraints are applied by constructing three additional equations for each contact node. Be different from the FEM, the contact equations are written explicitly with both tractions and displacements which are retained as unknowns in BEM. We apply the compatibility conditions to one of the bodies involved in the contact, and the equilibrium conditions to the other for establishing the complementary constraint equations which is possible to state that one body controls the displacements and the other controls the tractions.

In this section, we present two approaches to construct the contact equations for frictionless contact conditions. Before establishing the contact equations, we assume that body ***A*** controls the tractions, and body ***B*** controls the displacements on the contact zone. Therefore, the contact equations can be written as following:

- Equation for equilibrium condition on body ***B***

$$t_1^B = -\sum_{k=1}^{e} N_k(\xi,\eta)\, t_1^{k,A} \tag{13}$$

- Equation for compatibility of normal displacements on body ***A***

$$u_1^A = \sum_{k=1}^{e} N_k(\xi,\eta)\, u_1^{k,B} + \delta_1 \tag{14}$$

- Equation for frictionless condition

$$t_2^A = t_3^A = t_2^B = t_3^B = 0 \tag{15}$$

where the subscripts *1*, *2*, *3* refer to the local coordination axis shown as Fig.4. Different from the ‘node-to-node’ contact model in which *1*-axes is obtained by an average unit normal between the outward normal vectors at the two nodes, here, *1*-axes is taken coincident with the outward normal of the target element at the target point. The other two axes are taken in the plane perpendicular to the *1*-axes. δ_1 is the normal distance between the contact node and the target point. The contact equations are obtained by enforcing the equilibrium, compatibility and frictionless conditions at the contact nodes as shown in Table 1.

5. The solution procedure

In the numerical scheme presented here, a potential contact zone will be initially assumed in order to determine the final contact zone. The BIEs (Eq. (8)) and contact equations (as shown in section 4) are used to obtain the displacements and tractions at contact nodes. Then the contact conditions have to be checked and updated with the displacements and tractions at the contact nodes. An iterative procedure is used to obtain the correct solution with contact pressure (negative normal tractions) inside the contact zone and where no interpenetrations outside the contact zone. Meanwhile, the static condensation technique is used to obtain a system of a reduced order [36]. In the iterative procedure, only a part of unknowns is involved to solve the contact problems.

Therefore, the solution procedure to detect the correct contact zone has the following steps:

Step1. Definition of the initial contact zones $\boldsymbol{S_{CP}}$ on the contact bodies. A node is denoted as contact node when its projected distance to a corresponding boundary element is relatively small.

Step2. Calculating the boundary integrals in Eq. (8) and assembling the integral equations for the contact bodies.

Step3. Performing the static condensation procedure and assembling the final equation of system by introducing the boundary conditions and contact conditions.

Step4. Solving the final system and calculating the displacements and tractions at the contact nodes of the assumed $\boldsymbol{S_{CP}}$.

Step5. Checking the contact pressure at the contact nodes. If positive normal tractions are presented at some contact nodes, they are excluded from the contact zone, returning again to step 3. Otherwise, the algorithm follows to step 6.

Step6. Checking that no interpenetrations occur outside the contact zone. If there are interpenetrations, the nodes where interpenetrations have occurred are included in the contact zone, and returning again to step 3. Otherwise, the correct contact zone is obtained.

It is obvious that the computational efficiency of this algorithm depends to a great extent on the initial estimation of contact zone.

6. Numerical examples

In this section, several classical contact problems will be studied to illustrate the applicability of the presented method. In the first example, an elastic rectangular punch is compressed on the foundation. Since the contact zone is known, this problem can be solved in one step with the absence of friction. The second example presents an elastic curve punch compressing on the foundation. An iterative procedure has to be used to determine the final contact zone.

6.1 Compression of an elastic rectangular punch on foundation

This example corresponds to the contact problem as shown in Fig.6. The same properties have been taken for both bodies: Young's modulus E_p=E_b=200GPa and Poisson's ratio v=0.3. The uniform pressure p=2MPa is applied on the top face of punch. The bottom face of the foundation is fixed. Since the size of contact zone is known and cannot change, no iterative is needed for the solution.

In BFM, the discontinuous linear element has been used to discrete the boundaries of punch and foundation. As shown in Fig.7(a), a total of 600 elements have been obtained for the punch and 840 elements for the foundation. The static condensation technique is performed on both punch and foundation, where the variables corresponding to the nodes on the contact faces (the bottom face of punch and the top face of foundation) are involved. And only 1200 DOF and 1500 DOF are managed on punch and foundation, respectively, due to the static condensation.

In ABAQUS, the punch and foundation are both discretized by C3D8R element with mesh size 0.5mm. Therefore, 9261 nodes and 8000 elements on the punch and 35301 nodes and 32000 elements on the foundation are obtained as shown in Fig.7(b). The surface-to-surface discretization method and finite sliding formulation are used in the contact interaction. The comparison of the contact pressure along the red line which is the central axis of the bottom face of punch as shown in Fig.6(b) between ABAQUS and BFM is presented in Fig.8. The distribution of contact pressure on the contact zone is also shown in Fig.9. It can be seen that the contact pressure become very high as a singularity at the sharp edge of the punch and the BFM can get higher contact pressure to show the stress concentration.

Meanwhile, fixing the Young's modulus of the foundation, we change the Young's modulus of the punch and study the contact pressure along the red line with different relative stiffness E_p/E_b as shown in Fig.10.

When the punch becomes stiffer, the stress concentration phenomenon is more obvious. The contact pressure tends to a constant equal to the applied pressure with the relative stiffness E_p/E_b =0. As shown in Fig.10, the contact pressure is constant equal to the applied pressure p=2MPa, when the relative stiffness E_p/E_b is equal to 1/1000, where the foundation can be taken as the rigid body.

6.2 Compression of an elastic curve punch on foundation

As shown in Fig.11, an elastic curve punch is compressed on the foundation. The widths of the punch and foundation are both 25mm. The uniform pressure p=80KPa is applied on the top face of punch. The bottom face of the foundation is fixed. The same properties have been taken for both bodies: Young's modulus $\boldsymbol{E}$=2.1MPa and Poisson's ratio v=0.3. In this example with a unknown contact zone, an iterative algorithm is needed for determining the final contact zone.

In BFM, a total of 1508 elements have been obtained for the punch and 1528 elements for the foundation as shown in Fig.12(a). The static condensation technique is performed on both punch and foundation, where the variables corresponding to the nodes on the contact faces (the bottom face of punch and the top face of foundation) are involved. And only 3600 DOF and 4320 DOF are managed on punch and foundation, respectively, due to the static condensation. In ABAQUS, the punch and foundation are both discretized by C3D8R element with mesh size 1.2mm and 1.5mm, respectively. Therefore, 9792 nodes and 8415 elements on the punch and 10941 nodes and 9640 elements on the foundation are obtained as shown in Fig.12(b). The surface-to-surface discretization method and finite sliding formulation are used in the contact interaction.

The comparison of the contact pressure along the red line which is on the bottom face of punch as shown in Fig.11(b) between ABAQUS and BFM is presented in Fig.13. The distribution of contact pressure on the contact zone is also shown in Fig.14.

In Fig.13, it can be seen that the maximum contact pressure occurs at the center of the contact zone, and decreases steadily to zero at the edge of the contact zone. The contact pressure obtained by BFM is higher than that of ABAQUS.

6. Conclusions

Three-dimensional contact problems without friction have been studied using the Boundary Face Method (BFM). In the BFM, the numerical integrations of the boundary element are implemented in the parametric space of the boundary surfaces, where the geometric data at Gaussian quadrature points in integration elements, such as the coordinates, the Jacobian and the outward normal are calculated directly from the surface rather than elements, thus no geometric error will be introduced, and the unique outward normal on the contact nodes can be obtained by the parametric space of the boundary surface.

In this paper, a non-conforming contact discretization approach is used to enforce the contact conditions between the two contact surfaces. This method is based on Node-To-Surface (NTS), and there is no need that the identical discretization is performed along the contact surfaces of both bodies. The contact equations are written explicitly with both tractions and displacements which are retained as unknowns in BIE. An iterative procedure is presented to determine the correct contact zone by obtaining a solution compatible with the contact conditions (no interpenetrations between the domains and no tensile on the final contact zone). Several examples have been presented to demonstrate the applicability of the presented algorithm.

It is obvious that the solution procedure of the frictionless contact problems developed is relatively simple. However, it can be the first step in a frictional analysis which is our ongoing work.

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Table 1. Unknowns and corresponding equations for contact node

Unknowns at a contact node	Equations	
	Body ***A***	Body ***B***
u_1	Eq. (13)	BIE
u_2	BIE	BIE
u_3	BIE	BIE
t_1	BIE	Eq. (12)
t_2	Eq. (14)	Eq. (14)
t_3	Eq. (14)	Eq. (14)

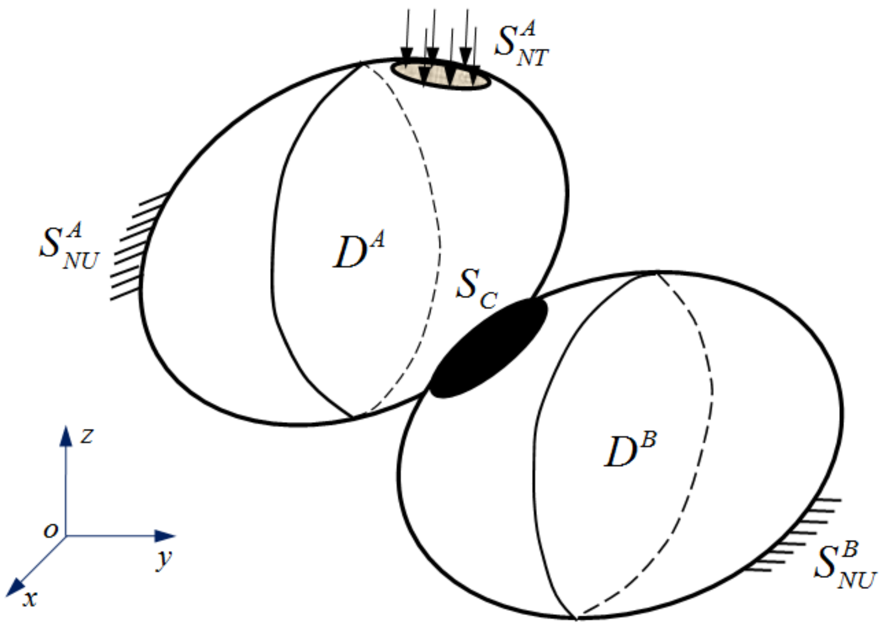


Fig.1. Problem definition

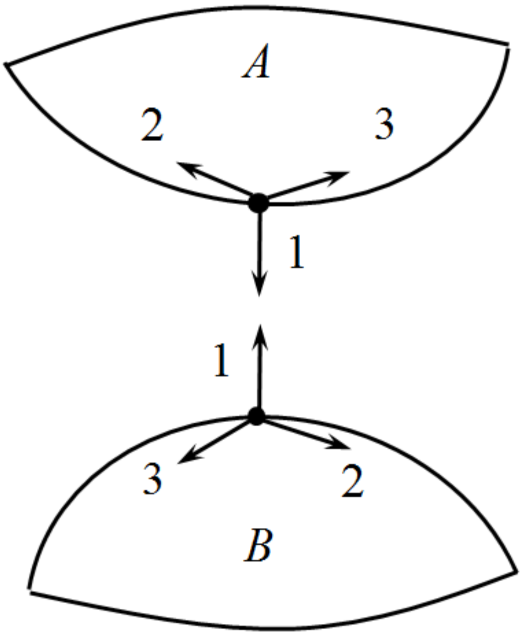
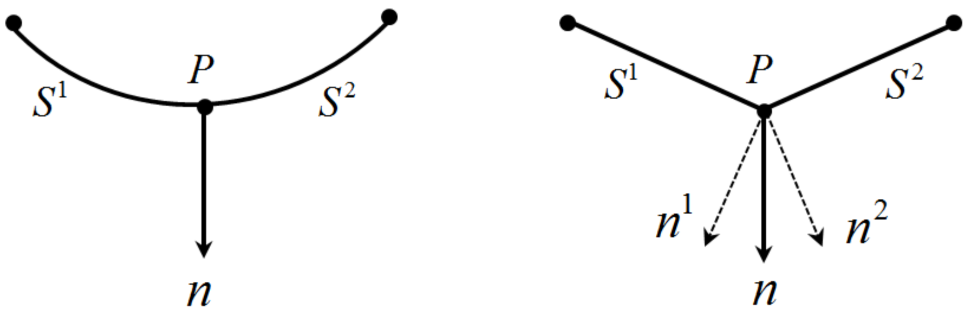


Fig.2. Two points in contact and the unit boundary vectors



(a) BFM (b) BEM/FEM

Fig.3. Definition of contact normal in BFM and BEM/FEM

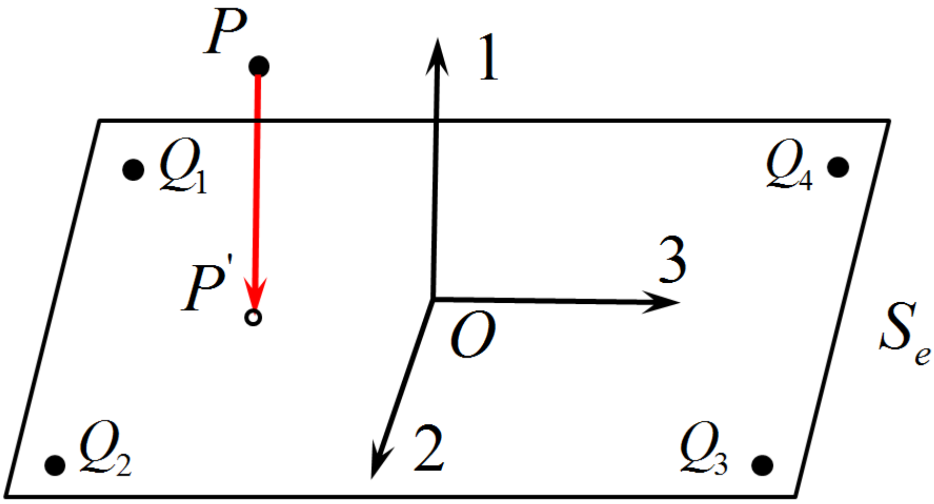


Fig.4. The ‘node-to-surface’ contact model

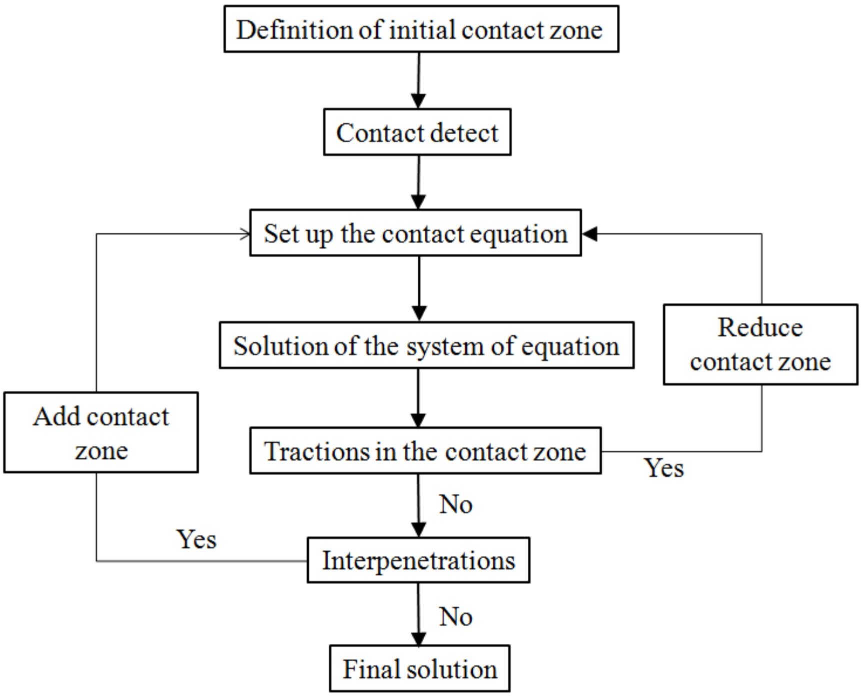
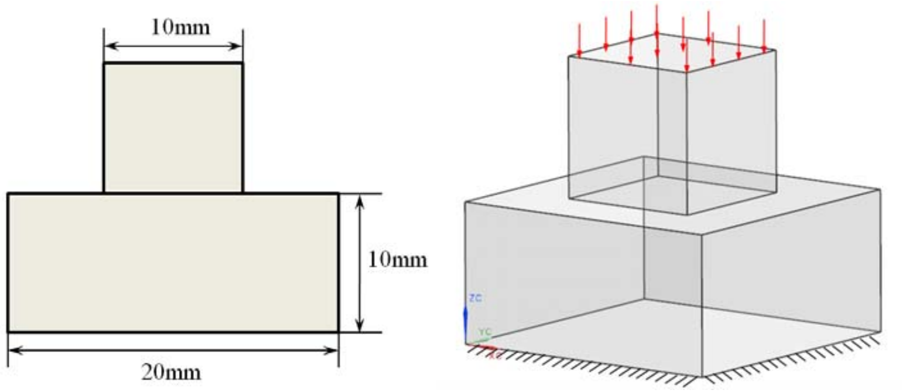
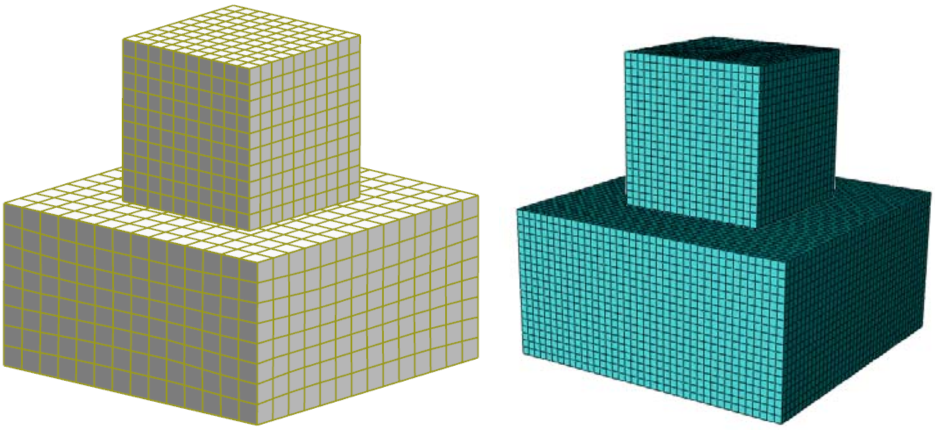


Fig.5. The iterative procedure for the frictionless contact problems



(a) Geometry (b) Problem definition

Fig.6. Elastic rectangular punch on foundation



(a) BFM (b) ABAQUS

Fig.7. Mesh models of BFM and ABAQUS

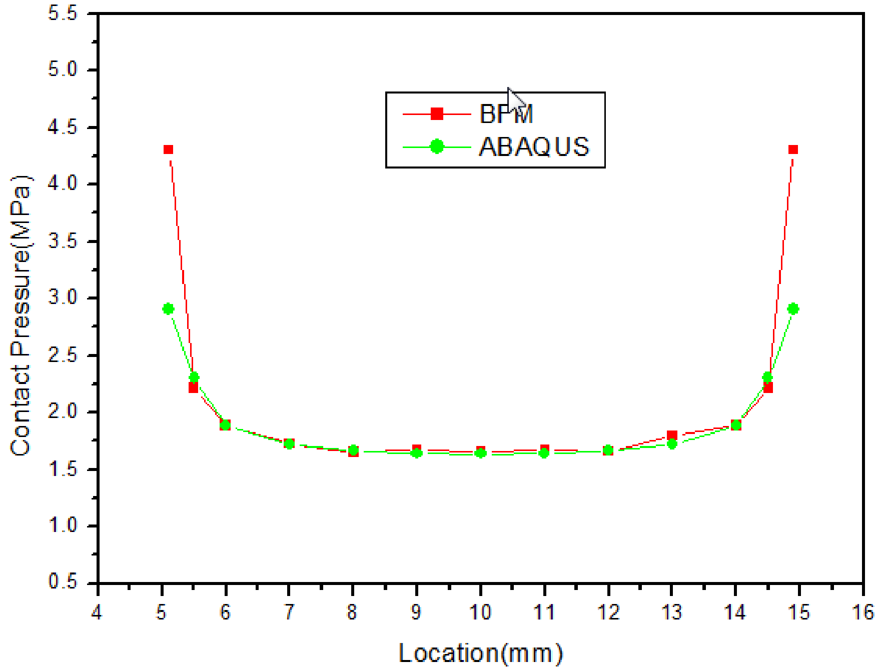
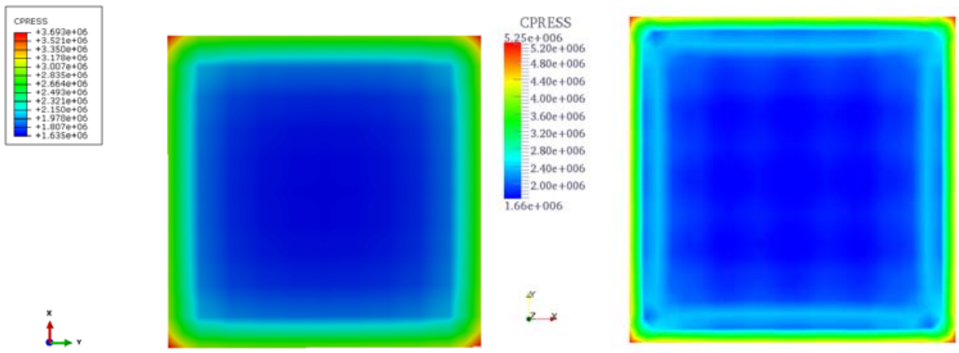


Fig.8. Comparison of the contact pressure between ABAQUS and BFM.



(a) ABAQUS (b) BFM

Fig.9. The distribution of contact pressure on the punch.

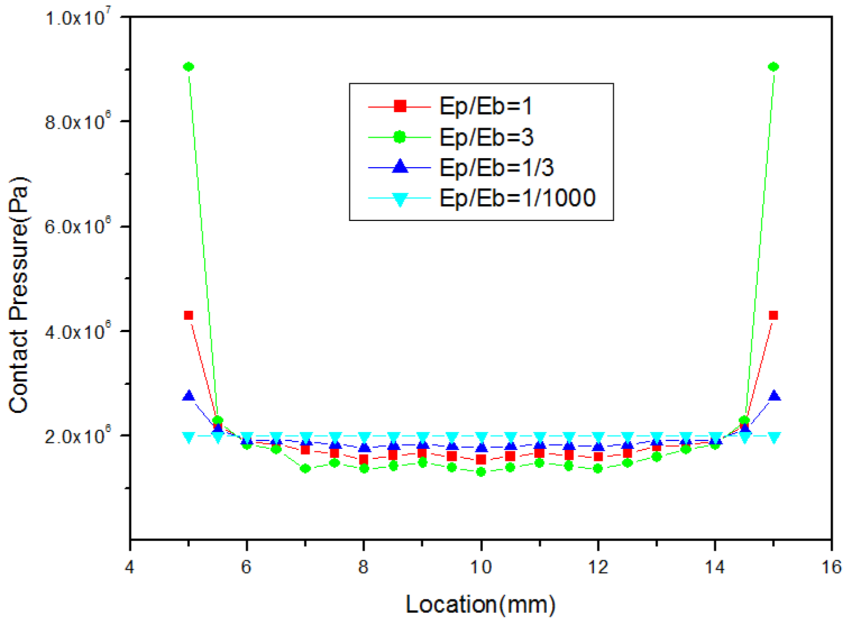
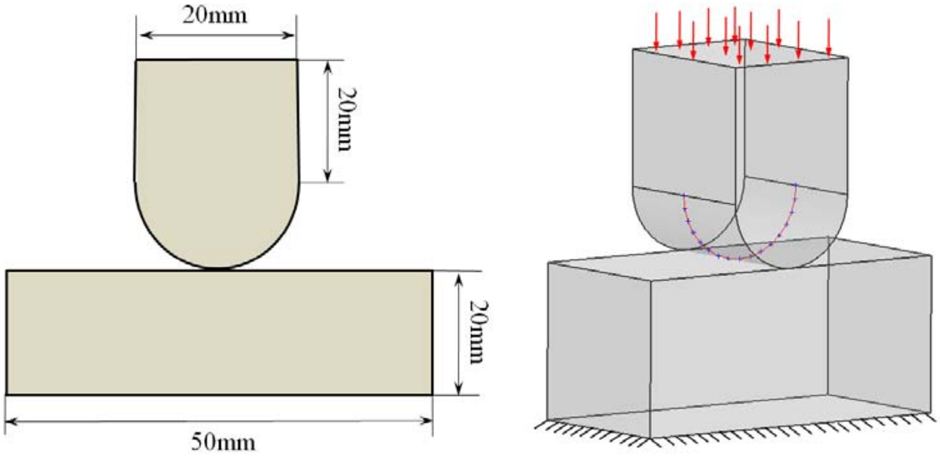
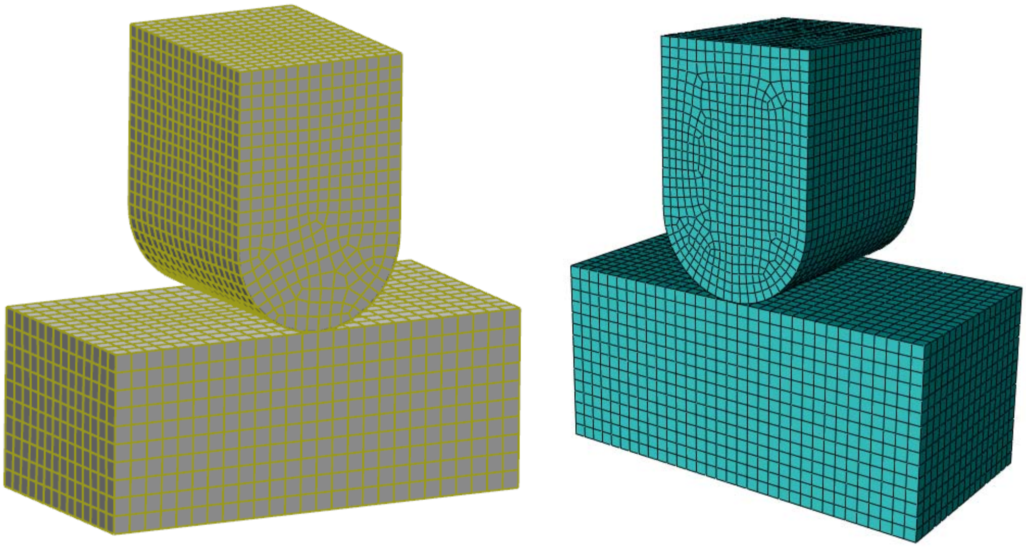


Fig.10. The contact pressure with different relative stiffness E_p/E_b.



(a) Geometry (b) Problem definition

Fig.11. Elastic curve punch on foundation.



(a) BFM (b) ABAQUS

Fig.12. Mesh models of BFM and ABAQUS.

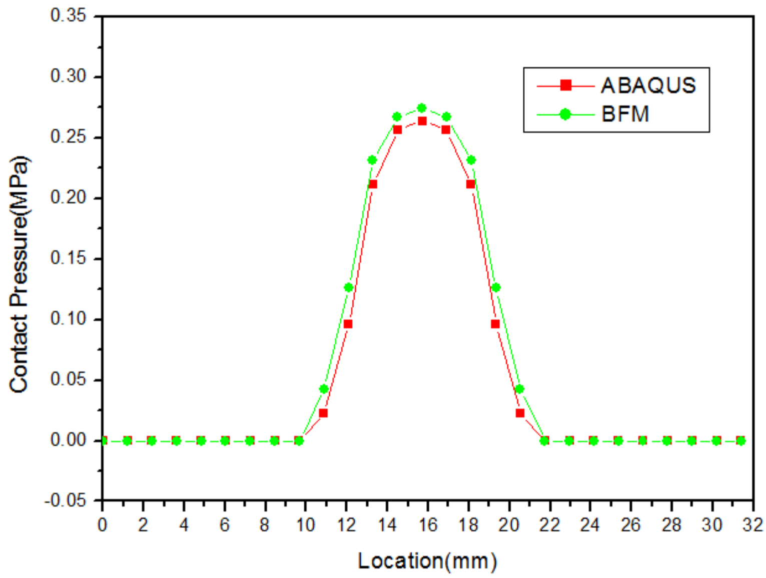


Fig.13. Comparison of the contact pressure between ABAQUS and BFM.

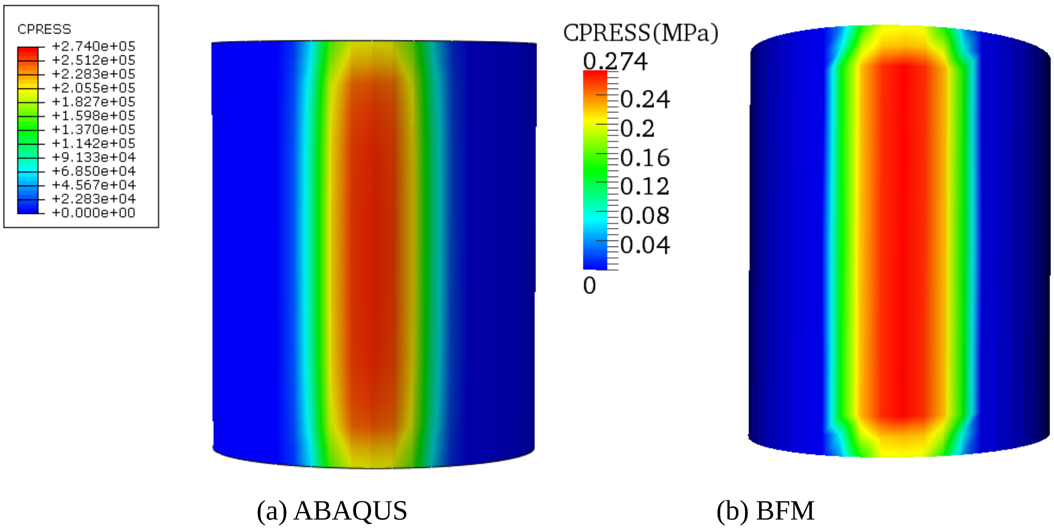


Fig.14. The distribution of contact pressure on the punch